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MASS OR HEAT TRANSFER FROM A SOLID BOUNDARY TO A TURBULENT FLUID

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Abstract—The problem of turbulent heat or mass transfer from a solid boundary is treated by dividing the turbulent flow into two regions. For the region in the immediate vicinity of the wall one uses a physical model proposed previously by the author. In the model one considers the turbulent motion as a succession along the wall of quasi-steady laminar motions each of them having a short path. Compared to the preceding treatment, in the present one takes into account various forms of dependence of the shear stress on the distance y to the wall. For a constant shear stress the case of the turbulent motion in a pipe is obtained, while for a shear stress proportional to y that of turbulent separated flows near the separation point. For the fully turbulent region, Prandtl's equation for the turbulent kinetic energy is used and on its basis one obtains an equation for the turbulent diffusion coefficient and for the mass transfer flux. The equation for the turbulent kinetic energy has allowed to propose a procedure for obtaining information concerning the length x_0 of the laminar path. The main conclusion is that the form of dependence of the shear stress on the distance from the wall has a great influence on the manner in which the mass transfer coefficient depends on the diffusion coefficient, viscosity and fluid viscosity.

NOMENCLATURE

a, universal constant;
A, quantity defined by equation (25);
b, universal constant;
B, quantity defined by equation (38);
c, concentration;
c_i, concentration at the interface;
c_0, concentration for $y = y_0$;
c_m, concentration in the bulk of the fluid;
d, pipe diameter;
D, diffusion coefficient;
e, e_1, e_2, universal constants;
E, kinetic turbulent energy, equation (27);
E_0, value of E for $y = y_0$;
f, friction factor;
G, universal constant, equation (46);
k, mass-transfer coefficient;
n, exponent;
N, mass flux;
$\overline{N}$, average value of N over the laminar path of length x_0;
P, pressure;
u, x component of velocity;
u_m, average velocity in a tube;
U, velocity at infinite of a fluid moving around a sphere or cylinder;
u', x component of velocity fluctuation;
v', y component of velocity fluctuation;
w', z component of velocity fluctuation;
$\overline{(u')^2}, \overline{(v')^2}, \overline{(w')^2}$, temporal average of $(u')^2$, $(v')^2$ and $(w')^2$;
x, distance along the wall;
x_0, the length of a laminar path;
y, distance from the wall;
y_0, thickness of the layer from the immediate vicinity of the wall;
y_1, distance at which $c \approx c_m$;
α, quantity defined by equations (3) or (4);
β, proportionality constant taken equal to $3 + n$;

Γ, gamma function;
δ, length in the similarity variable μ, equation (A.1); and given by (A.4);
ε, turbulent diffusion coefficient;
η, dynamic viscosity;
μ, similarity variable, defined by equation (A.1) and given by equation (11);
ν, kinematic viscosity;
τ, shear stress;
τ_0, value of τ at the wall;
ρ, fluid density;
Re, $u_m d/\nu$;
Sc, ν/D.

INTRODUCTION

SOME years ago a physical model was developed by the author for representing the turbulent heat or mass transfer near a solid boundary for large Schmidt or Prandtl numbers [1, 2]. Owing to some assumptions, that treatment is however valid only if the shear stress near the boundary is practically independent of the distance y from the solid surface, condition fulfilled, for instance, for the flow in a tube. The model was extended recently to turbulent separated flows [3]. In these cases the shear stress near the solid boundary is no longer independent of y. Spalding's paper [4] concerning the heat transfer from turbulent separated flows has stimulated us to look for a more unitary analysis of the cases treated in [1] and [3], and has also provided means for solving the problem for a larger range of Schmidt or Prandtl numbers. It is the aim of the present paper to present the more general approach of the turbulent heat or mass transfer from a solid boundary for a large range of Schmidt or Prandtl numbers. In the analysis, the turbulent flow near a solid wall will be divided into two regions. One of them in the immediate vicinity of the wall in which the molecular diffusion coefficient and the molecular viscosity are acting and the other one, the fully turbulent region, in which the effect of the molecular physical quantities is negligible. For the region in the immediate vicinity of the wall the author's physical model will be used, while for the fully turbulent region the equation written by Prandtl [5] for the turbulent kinetic energy and applied by Spalding to turbulent separated flows [4]. Compared to the previous models [1, 3], the present one leads to a more general method owing both to the manner in which one takes into account the dependence of the shear stress on the distance to the wall and to the manner in which equations are established for some hydrodynamic quantities introduced by the model. The use of Prandtl's equation for the turbulent kinetic energy in the fully turbulent region permits to obtain equations for the mass or heat-transfer coefficient valid for a large range of Schmidt or Prandtl numbers.

THE REGION IN THE IMMEDIATE VICINITY OF THE WALL

In the immediate vicinity of the wall it will be considered that the turbulent motion is composed of a succession of laminar motions along the wall, each of them having a short path of length x_0 [1, 2]. Owing to turbulent fluctuations, elements of liquid brought to the wall, are moving along the wall in short paths of length x_0 and are dissolving into the bulk of the fluid, being replaced by other elements of fluid and so on. Arguments in support of this model based especially on visual observations, but also on an analysis of the turbulent diffusion coefficient concept [6], were adduced in previous papers and will not be repeated here. The equations of motion and the convective diffusion equation valid for each path are those for the corresponding quasi-steady laminar motion. For simplicity a one-dimensional distribution will be taken for the velocity field in each path. This field will be obtained from the equation

$$\tau = \eta \frac{\mathrm{d}u}{\mathrm{d}y}. \quad (1)$$

For the turbulent motion in a tube

$$\tau \approx \tau_0 = \text{constant} \tag{2}$$

near the wall, while for turbulent separated flows the shear stress is small at the wall near the separation point (being nil at this point) and depends on the distance y from the wall. In the last case it is possible to write

$$\tau = \tau_0 + y\alpha. \tag{3}$$

The validity conditions of (3) are discussed by Townsend [7]. The quantity α may be taken approximately equal to dP/dx. The shear stress τ_0 at the wall is positive before the separation point and negative behind the separation point. In what follows, for simplicity, only the process taking place in a region very near to the separation point where $\tau_0 \approx 0$ will be considered.

Owing to the fact that the cases in which τ has the form

$$\tau = \alpha y^n \tag{4}$$

may be solved easily and to the fact that (4) includes both the case of a tube ($n = 0$) and that of separated flows at the separation point ($n = 1$), the treatment will be based on it.

Introducing (4) into (1) and integrating one gets

$$u = \frac{\alpha}{\eta(1+n)} y^{n+1}. \tag{5}$$

The convective diffusion equation valid in each path takes the form

$$\frac{\alpha}{(1+n)\eta} y^{n+1} \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2}. \tag{6}$$

If the assumption is made that the thickness of the fluid elements moving along the wall (and in contact with it) a distance equal to x_0 is larger than the depths of penetration by diffusion for all values of $x \leqslant x_0$, one may approximate the distribution of concentration by the one valid for a semi-infinite fluid. Consequently equation (6) must be solved for the boundary conditions

$$c = c_i \quad \text{for} \quad y = 0 \tag{7}$$

$$c = c_0 \quad \text{for} \quad x = 0 \tag{8}$$

$$c = c_0 \quad \text{for} \quad y \to \infty. \tag{9}$$

The solution of equation (6) for the boundary conditions (7)–(9) has the form (see appendix)

$$\frac{c - c_i}{c_0 - c_i} = \frac{\int_0^\mu \exp(-s^{n+3})\,ds}{\Gamma\left(\dfrac{n+4}{n+3}\right)} \tag{10}$$

where

$$\mu \equiv \frac{y}{\left(\dfrac{(n+3)^2(n+1)\eta}{\alpha}\right)^{\frac{1}{n+3}} (Dx)^{\frac{1}{n+3}}}. \tag{11}$$

For the mass flux one obtains

$$N = -D\left(\frac{\partial c}{\partial y}\right)_{y=0} = \frac{D(c_i - c_0)}{\Gamma\left(\dfrac{n+4}{n+3}\right)} \times \left(\frac{\alpha}{(n+3)^2(n+1)\eta}\right)^{\frac{1}{n+3}} (Dx)^{-\frac{1}{n+3}} \tag{12}$$

and for the average mass flux defined over the path of length x_0

$$\bar{N} = \frac{n+3}{n+2} \frac{D^{\frac{n+2}{n+3}}(c_i - c_0)}{\Gamma\left(\dfrac{n+4}{n+3}\right)} \times \left(\frac{\alpha}{(n+3)^2(n+1)\eta}\right)^{\frac{1}{n+3}} x_0^{-\frac{1}{n+3}}. \tag{13}$$

The length x_0 of the path being short one may consider $\bar{N}$ as a quasi-local quantity.

For large values of Schmidt number the concentration c_0 is practically equal to that of the bulk and for the mass-transfer coefficient, defined by

$$k = \bar{N}/(c_i - c_m), \tag{14}$$

one obtains

$$k = \frac{n+3}{n+2} \frac{D^{\frac{n+2}{n+3}}}{\Gamma\left(\dfrac{n+4}{n+3}\right)} \times \left(\frac{\alpha}{(n+3)^2(n+1)\eta}\right)^{\frac{1}{n+3}} x_0^{-\frac{1}{n+3}}. \tag{15}$$

The transfer coefficient may be also considered a quasi-local quantity.

The length x_0 of the laminar path may be obtained by means of dimensional considerations. For a tube one considers that the state of turbulence near the boundary is characterized by τ_0, ρ and η. Extending this type of reasoning to the present case one may say that the state of turbulence is characterized by α, ρ and η: Dimensional considerations lead to

$$x_0 \propto \left(\frac{\alpha}{\eta}\right)^{-\frac{1}{n+2}} \nu^{\frac{1}{n+2}} \tag{16}$$

Consequently

$$\bar{N} \propto D^{\frac{n+2}{n+3}} \left(\frac{\alpha}{\rho}\right)^{\frac{1}{n+2}} \nu^{-\frac{n+4}{(n+2)(n+3)}} (c_i - c_0) \tag{17}$$

and the mass-transfer coefficient for large Schmidt numbers is given by

$$k \propto D^{\frac{n+2}{n+3}} \left(\frac{\alpha}{\rho}\right)^{\frac{1}{n+2}} \nu^{-\frac{n+4}{(n+2)(n+3)}}. \tag{18}$$

For $n = 0$, equation (18) leads to that obtained in [1], and for $n = 1$ to that obtained in [3]. One may stress the fact that a similar equation may be obtained by means of the turbulent diffusion coefficient concept if the procedure from [3] is used.

The dimensional analysis has led to an equation for x_0 owing to the fact that the number of physical quantities implied is four (x_0, α, ρ, η). Such considerations cannot lead however to information about the dependence of the proportionality constant from (16) on n, and to equations for x_0 corresponding to more complicated relations between τ and y, as is for instance that given by equation (3). A more satisfactory procedure for obtaining an equation for the length x_0 of the laminar path is therefore needed. An analysis of the problem in terms of more fundamental (primary) quantities may be useful in this respect. The eddies having as a consequence the succession of short laminar motions along the wall are generated in the vicinity of the boundary between the two considered regions. The intensity of the eddies may be characterized by their kinetic energy E_0 at this point. It is natural to consider that x_0 depends on E_0 and on the physical constants η and ρ. Dimensional considerations permit us to write

$$x_0 = e_1 \frac{\nu}{E_0^{\frac{1}{2}}} \tag{19}$$

where e_1 is an universal constant.

Though equation (19) was also deduced by dimensional reasoning, it is more general than (16) since it does not contain particular quantities specific to a certain case. It is however, necessary to express, for each situation, E_0 as a function of the corresponding particular quantities. It will be shown in the following section that for the fully turbulent region it is possible to establish a differential equation for the dependence of E on the distance y, equation in which the shear stress τ appears too. For each type of dependence of τ on y results a dependence of E on y. Since for the value of y at the boundary of the two regions dimensional considerations lead to

$$y_0 = \frac{e_2 \nu}{E_0^{\frac{1}{2}}}, \tag{20}$$

the above mentioned equation $E = E(y)$ permits us to establish an equation for E_0.

For instance, for $\tau = \alpha y^n$, one obtains (see equation (32) from the following section)

$$E = \frac{\alpha/\rho}{[e(a - \frac{3}{2}bn^2)]^{\frac{1}{2}}} y^n. \tag{21}$$

Because

$$E_0 = \frac{\alpha/\rho}{[e(a - \frac{3}{2}bn^2)]^{\frac{1}{2}}} y_0^n, \tag{22}$$

using equation (20), one gets

$$E_0 = \left\{\frac{e_2^n}{[e(a - \frac{3}{2}bn^2)]^{\frac{1}{2}}}\right\}^{\frac{2}{2+n}} \nu^{\frac{2n}{2+n}} \left(\frac{\alpha}{\rho}\right)^{\frac{2}{2+n}}. \tag{23}$$

Consequently

$$x_0 = e_1 \frac{[e(a - \frac{3}{2}bn^2)]^{\frac{1}{2(2+n)}}}{e_2^{\frac{n}{2+n}}} \nu^{\frac{2}{2+n}} \left(\frac{\alpha}{\rho}\right)^{-\frac{1}{2+n}}. \tag{24}$$

Compared to equation (16), equation (24) takes into account the dependence on the exponent n too.

For the average mass flux taken over the laminar path of length x_0, one obtains

$$\bar{N} = \frac{n+3}{n+2} \frac{e_1^{-\frac{1}{n+3}} \left[e\left(a - \tfrac{3}{2} bn^2\right)\right]^{-\frac{1}{2(n+2)(n+3)}}}{\left[(n+1)(n+3)^2\right]^{\frac{1}{n+3}} e_2^{-\frac{n}{(n+3)(n+2)}} \Gamma\left(\frac{n+4}{n+3}\right)}$$

$$\times D^{\frac{n+2}{n+3}} \left(\frac{\alpha}{\rho}\right)^{\frac{1}{n+2}} \nu^{-\frac{n+4}{(n+2)(n+3)}} \times (c_i - c_0)$$

$$\equiv A \left(\frac{\alpha}{\rho}\right)^{\frac{1}{n+2}} D^{\frac{n+2}{n+3}} \nu^{-\frac{n+4}{(n+2)(n+3)}} (c_i - c_0), \quad (25)$$

where the dimensionless quantity A is a function of the exponent n.

The mass-transfer coefficient at large Schmidt numbers is therefore given by

$$k = A \left(\frac{\alpha}{\rho}\right)^{\frac{1}{n+2}} \nu^{-\frac{n+4}{(n+2)(n+3)}} D^{\frac{n+2}{n+3}}. \quad (26)$$

THE FULLY TURBULENT REGION

The method of analysis which will be used in the following was developed by Prandtl [5] and was applied recently by Spalding [4] to turbulent separated flows. As in Spalding's treatment, for the sake of simplicity, a one-dimensional equation will be used for the description of turbulence in this region. This is a rough approximation for turbulent separated flows. Owing to the dependence of τ_0 and α on x, it must be described by at least a two-dimensional equation. The one dimensional approximation is probably satisfactory in a small zone near the separation point (where $\tau_0 \approx 0$ and α is practically constant). This is the only zone considered in the present paper.

The main quantity appearing in Prandtl's equation [5] is the turbulent kinetic energy E

$$E \equiv \tfrac{1}{2}\left[\overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2}\right] \quad (27)$$

An equation for E results by equating the rate of dissipation per unit volume of the turbulent energy with the sum of the variation of the diffusion flux of turbulent energy and of the rate of generation per unit volume of turbulent energy.

Expressions for the rate of dissipation per unit volume and for the turbulent diffusion coefficient ε are obtained by means of dimensional analysis:

$$\text{rate of dissipation per unit volume} = \frac{a\rho E^{\frac{3}{2}}}{y} \quad (28a)$$

$$\varepsilon = eE^{\frac{1}{2}} y, \quad (28b)$$

a and e being universal constants.

Since the rate of generation is given by $\tau\, \mathrm{d}u/\mathrm{d}y$, while the variation of the diffusion flux of the turbulent energy by $b\rho\, \mathrm{d}/\mathrm{d}y\, (E^{\frac{1}{2}} y\, \mathrm{d}E/\mathrm{d}y)$ one gets

$$\frac{aE^{\frac{3}{2}}}{y} - b \frac{\mathrm{d}}{\mathrm{d}y}\left(yE^{\frac{1}{2}} \frac{\mathrm{d}E}{\mathrm{d}y}\right) - \frac{\tau}{\rho} \frac{\mathrm{d}u}{\mathrm{d}y} = 0. \quad (29)$$

Eliminating $\mathrm{d}u/\mathrm{d}y$ by means of the equation

$$\frac{\tau}{\rho} = eE^{\frac{1}{2}} y \frac{\mathrm{d}u}{\mathrm{d}y} \quad (30)$$

and using equation (4) for τ/ρ it results

$$\frac{aE^{\frac{3}{2}}}{y} - b \frac{\mathrm{d}}{\mathrm{d}y}\left(yE^{\frac{1}{2}} \frac{\mathrm{d}E}{\mathrm{d}y}\right) - \frac{\alpha^2 y^{2n}}{e\rho^2 E^{\frac{1}{2}} y} = 0. \quad (31)$$

The solution of equation (31) which satisfies the boundary condition

$$E \to 0 \quad \text{for} \quad y = 0$$

has the form

$$E = \frac{\alpha/\rho}{\left[e\left(a - \tfrac{3}{2} bn^2\right)\right]^{\frac{1}{2}}} y^n. \quad (32)$$

Consequently the turbulent diffusion coefficient is given by

$$\varepsilon = e \frac{(\alpha/\rho)^{\frac{1}{2}}}{\left[e\left(a - \tfrac{3}{2} bn^2\right)\right]^{\frac{1}{4}}} y^{1+\frac{n}{2}}. \quad (33)$$

For the mass flux in the fully turbulent region one may write

$$N = -\varepsilon \frac{\mathrm{d}c}{\mathrm{d}y} \quad (34)$$

By integration one gets

$$\frac{c_0 - c_m}{N} = \frac{2[e(a - \frac{3}{2}bn^2)]^{\frac{1}{2}}}{ne(\alpha/\rho)^{\frac{1}{2}}} \times \left(\frac{1}{y_0^{\frac{n}{2}}} - \frac{1}{y_1^{\frac{n}{2}}}\right) \quad \text{for} \quad n \neq 0. \tag{35}$$

Considering y_1 sufficiently large and n not too near to zero equation (35) becomes

$$\frac{c_0 - c_m}{N} = \frac{2}{ne}[e(a - \tfrac{3}{2}bn^2)]^{\frac{1}{2}}(\alpha/\rho)^{-\frac{1}{2}} y_0^{-\frac{n}{2}}. \tag{36}$$

It must be stressed that equation (36) is not valid for turbulent flow in a pipe ($n = 0$). This case was examined previously [2, 10].

For the thickness y_0 of the region from the immediate vicinity of the wall, equation (20) together with the equation

$$E_0 = \frac{\alpha/\rho}{[e(a - \frac{3}{2}bn^2)]^{\frac{1}{2}}} y_0^n$$

which results from (32), lead to

$$y_0 = e_2^{\frac{2}{2+n}}[e(a - \tfrac{3}{2}bn^2)]^{\frac{1}{2(2+n)}}\left(\frac{\alpha}{\rho}\right)^{-\frac{1}{2+n}}. \tag{37}$$

Consequently the mass flux in the fully turbulent region is given by the equation:

$$N = \frac{ne}{2} e_2^{\frac{n}{2+n}}[e(a - \tfrac{3}{2}bn^2)]^{\frac{1}{2(2+n)}} \times \nu^{\frac{n}{2+n}}\left(\frac{\alpha}{\rho}\right)^{\frac{1}{2+n}}(c_0 - c_m) \equiv B\nu^{\frac{n}{2+n}} \times \left(\frac{\alpha}{\rho}\right)^{\frac{1}{n+2}}(c_0 - c_m), \tag{38}$$

where the dimensionless quantity B depends on n.

THE MASS-TRANSFER COEFFICIENT

Equations (25) and (38) may be written under the form

$$N = \frac{c_i - c_0}{\frac{1}{A} D^{-\frac{n+2}{n+3}}\left(\frac{\alpha}{\rho}\right)^{-\frac{1}{n+2}} \nu^{\frac{n+4}{(n+2)(n+3)}}} = \frac{c_0 - c_m}{\frac{1}{B}\nu^{-\frac{n}{2+n}}\left(\frac{\alpha}{\rho}\right)^{-\frac{1}{n+2}}} = \frac{1}{\frac{1}{A}\left(\frac{\nu}{D}\right)^{\frac{n+2}{n+3}} + \frac{1}{B}} \times \left(\frac{\alpha}{\rho}\right)^{\frac{1}{n+2}} \nu^{\frac{n}{2+n}}(c_i - c_m). \tag{39}$$

The mass-transfer coefficient is thus given by

$$k = \frac{\left(\frac{\alpha}{\rho}\right)^{\frac{1}{n+2}} \nu^{\frac{n}{2+n}}}{\frac{1}{A}\left(\frac{\nu}{D}\right)^{\frac{n+2}{n+3}} + \frac{1}{B}}. \tag{40}$$

Equation (40) represents the main result of the present analysis.

DISCUSSION

Two equations have been established for the mass-transfer coefficient. One of them, equation (26), for large Schmidt numbers and the other one, equation (40), for a larger range of Schmidt numbers. The fact must be, however, stressed that the first equation is valid for any value of $n \geqslant 0$, while the second only for those values which are not near zero. The first comment which may be made with respect to the mentioned equations is that the manner in which the shear stress depends on y has a great effect on the dependence of the mass-transfer coefficient on the diffusion coefficient, viscosity and velocity. For large Schmidt numbers

$$k \propto D^{\frac{2}{3}} \nu^{-\frac{2}{3}}\left(\frac{\tau}{\rho}\right)^{\frac{1}{2}} \quad \text{for } n = 0 \tag{41}$$

$$k \propto D^{\frac{3}{4}} \nu^{-\frac{5}{12}}\left(\frac{\alpha}{\rho}\right)^{\frac{1}{3}} \quad \text{for } n = 1. \tag{42}$$

Because for a pipe $\tau_0/\rho \propto Re^{-0\cdot 2}\, u_m^2$, equation (41) leads to

$$k \propto D^{\frac{2}{3}}\, \nu^{-\frac{2}{3}}\, Re^{-0\cdot 1}\, u_m \quad \text{for } n = 0. \qquad (43)$$

Considering that in the vicinity of the separation point

$$\frac{\mathrm{d}P}{\mathrm{d}x} \propto U^2,$$

equation (42) becomes

$$k \propto D^{\frac{3}{4}}\, \nu^{-\frac{5}{12}}\, U^{\frac{2}{3}} \qquad \text{for} \quad n = 1. \qquad (44)$$

A comparison between equation (43) and (44) shows that the exponent of the diffusion coefficient is $\frac{2}{3}$ in the first case and $\frac{3}{4}$ in the second and that the exponent of velocity is 0·9 in the first case and $\frac{2}{3}$ in the second. The experiments performed by Harriot and Hamilton [8] for the dissolving of the wall of a pipe into the turbulent liquid flowing inside the pipe and for a large range of Schmidt number (between 4×10^2 up to 10^5) lead to exponents very near to the ones obtained above

$$k \propto D^{0\cdot 654}\, U^{0\cdot 887} \qquad (45)$$

Richardson [9] studying the heat and mass transfer in turbulent separated flows has obtained experimentally a value of $\frac{2}{3}$ for the exponent of velocity. The same exponent, linked however to the assumption that $\mathrm{d}p/\mathrm{d}x \propto U^2$, appears in the present equation (44) too. Since for α no detailed information exists, it is not yet possible to compare in more details the obtained equation with experiment.

As concerns the equation (40), valid for a larger range of Prandtl number, it is of interest to note that for $n = 1$ the dependence on the hydrodynamic parameters is not changed by the Schmidt numbers, while in the case of a tube (equation (40) is not valid in this case) there exists such an interaction. For a tube it was shown previously [10] that

$$\frac{k}{u_m} = \frac{f/2}{1 + G[\sqrt{(f/2)}](Sc^{\frac{2}{3}} - 1)}. \qquad (46)$$

As was stressed above, the present analysis is valid only in the vicinity of the separation point. For a more complete analysis of the separated turbulent flows it is, however, necessary to use for τ equation (3) and at least a two-dimensional description of the fully turbulent field.

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APPENDIX

Equation (6) must be solved for the boundary conditions (7)–(9). The similarity variable

$$\mu = \frac{y}{\delta(x)} \qquad (A.1)$$

allows to transform equation (6) into

$$D\frac{\mathrm{d}^2 c}{\mathrm{d}\mu^2} + \frac{\alpha}{(1+n)\eta}\left(\frac{y}{\delta}\right)^{n+2} \delta^{n+2}\frac{\mathrm{d}\delta}{\mathrm{d}x}\frac{\mathrm{d}c}{\mathrm{d}\mu} = 0. \qquad (A.2)$$

The assumption that $c = c(\mu)$ is compatible with equation (A.2) if

$$\frac{\alpha}{(1+n)\eta}\,\delta^{n+2}\frac{\mathrm{d}\delta}{\mathrm{d}x} = \beta D. \qquad (A.3)$$

It is convenient to select for the constant β the value $\beta = n + 3$. From equation (A.3) one obtains

$$\delta = \left(\frac{(n+3)^2(n+1)\eta}{\alpha}\right)^{\frac{1}{n+3}}(Dx)^{\frac{1}{n+3}} \qquad (A.4)$$

Equation (A.2) becomes

$$\frac{\mathrm{d}^2 c}{\mathrm{d}\mu^2} + (n+3)\,\mu^{n+2}\frac{\mathrm{d}c}{\mathrm{d}\mu} = 0. \qquad (A.5)$$

Integrating and taking into account the boundary conditions one gets

$$\frac{c - c_i}{c_0 - c_i} = \frac{\int_0^{\mu} \exp(-s^{n+3})\,ds}{\int_0^{\infty} \exp(-s^{n+3})\,ds}. \tag{A.6}$$

Since

$$\int_0^{\infty} \exp(-s^{n+3})\,ds = \Gamma\left(\frac{n+4}{n+3}\right) \tag{A.7}$$

equation (A.6) leads to equation (10).

TRANSPORT DE MASSE OU DE CHALEUR À PARTIR D'UNE FRONTIÈRE SOLIDE VERS UN FLUIDE TURBULENT

Résumé—Le problème de transport de chaleur ou de masse à partir d'une frontière solide est traité en divisant l'écoulement turbulent en deux régions. Pour la région dans le voisinage immédiat de la paroi, on emploie un modèle physique proposé auparavant par l'auteur. Dans le modèle, on sonsidère le mouvement turbulent comme une succession le long de la paroi de mouvements laminaires quasi-permanents, chacun ayant un faible parcours. A la différence du traitement précédent, on tient compte dans le traitement actuel, de différentes formes de d'épendances de la contrainte de cisaillement en fonction de la distance y à la paroi. Pour une contrainte de cisaillement constante, on obtient le cas du mouvement turbulent dans un tube, tandis que, pour une contrainte de cisaillement proportionnelle à y, on obtient celui des écoulements turbulents décollés près du point de décollement. Pour la région entièrement turbulents, l'équation de Prandtl pour l'énergie cinétique turbulente est employée et, en se basant sur elle, on obtient une équation pour le coefficient de diffusion turbulent et pour le flux de transport de masse. L'équation pour l'énergie cinétique turbulente a permis de proposer un processus pour obtenir des informations concernant la longueur x_0 du parcours laminaire. La conclusion principale est que la forme de dépendance de la contrainte de cisaillement en fonction de la distance à la paroi à une grande influence sur la manière selon laquelle le coefficient de transport de masse dépend du coefficient de diffusion, de la viscosité et de la vitesse du fluide.

STOFF- ODER WÄRMEÜBERTRAGUNG VON EINER FESTEN GRENZFLÄCHE AN EINE TURBULENTE FLÜSSIGKEIT

Zusammenfassung—Das Problem der turbulenten Wärme- oder Stoffübertragung von einer festen Grenzfläche wird durch Aufteilung der turbulenten Strömung in zwei Bereiche behandelt. Für den Bereich in unmittelbarer Nähe der Wand verwendet man ein physikalisches Modell, das vor kurzem vom Autor vorgeschlagen wurde. Bei dem Modell betrachtet man die turbulente Bewegung als einer Aufeinanderfolge quasistationärer laminarer Bewegungen entlang der Wand, deren jede einen kurzen Weg zurücklegt. Im Vergleich mit der vorausgehenden Behandlung zieht man bei der gegenwärtigen verschiedene Formen der Abhängigkeit der Schubspannung vom Wandabstand y in Betracht. Bei konstanter Schubspannung erhält man den Fall der turbulenten Bewegung in einem Rohr, für eine Schubspannung proportional zu y jenen turbulenter abgelöster Strömungen nahe dem Ablösepunkt. Für den voll turbulenten Bereich wird Prandtl's Gleichung für die turbulente kinetische Energie benutzt. Auf Ihrer Grundlage erhält man eine Gleichung für den turbulenten Diffusionskoeffizienten und für den Stoffübertragungstrom. Die Gleichung für die turbulente kinetische Energie erlaubt es, ein Verfahren vorzuschlagen, um eine Auskunft bezüglich der Länge x_0 des laminaren Weges zu erhalten. Die hauptsächliche Schlussfolgerung ist, dass die Vorm der Abhängigkeit der Schubspannung vom Wandabstand einen grossen Einfluss auf die Art hat, wie der Stoffübertragungskoeffizient vom Diffusionskoeffizienten, der Zähigkeit und der Geschwindigkeit der Flüssigkeit abhängt.

ТЕПЛО—И МАССООБМЕН МЕЖДУ ТВЕРДОЙ ГРАНИЦЕЙ И ТУРБУЛЕНТНЫМ ПОТОКОМ ЖИДКОСТИ

Аннотация—Задача о переносе тепла (массы) с твёрдой поверхности рассматривается с помощью разделения турбулентного течения на две зоны. Для зоны в непосредственной близости от стенки используется модель, ранее предложенная автором. В модели турбулентное движение рассматривается как непрерывный ряд квазистационарных ламинарных течений вдоль стенки, причём каждое из них имеет своё «время жизни». По сравнению с предыдущими трактовками в настоящей принимаются во внимание формы зависимости сдвигового напряжения от расстояния у стенки. Найдено, что для случая постоянного сдвигового напряжения рассматриваемая модель соответствует турбулентному течению в трубе, тогда как для сдвигового напряжения, пропорциональ ного расстоянию y, течение имеет вид турбулентных отрывных потоков возле точки отрыва. Для области полностью развитого турбулентного течения используется уравнение для турбулентной кинетической энергии Прандтля, и на его основе получено уравнение для коэффициента турбулентной диффузии и потока переноса массы. Уравнение турбулентной кинетической энергии позволило предложить метод получения информации относительно длины X_0 ламинарной области. Основной вывод состоит в том, что форма зависимости сдвигового напряжения от расстояния до стенки оказывает большое влияние на вид зависимости коэффициента теплообмена от коэффициента диффузии, вязкости и скорости жидкости.